$\mathcal{D}$-Geometric Hilbert and Quot DG-Schemes

(Derived Hilbert scheme of solutions to Nonlinear PDE)

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Introduction

Goal

Study global (virtual) geometry of (derived) enhancements of moduli spaces of solutions to non-linear PDEs and their deformations (+ enumerative problems, DT/GW/PT-theory, Mirror symmetry ...).

Based on arXiv:2507.07937 and arXiv:2411.02387 (with J.Kryczka) and previous work with J. Kryczka-A.S and S-T.Yau:

- **Derived Moduli of Solutions to (Nonlinear) PDEs**: Part I [KSY] and Part II [KSY2] (Kryczka-S-Yau)
- **Hilbert and Quot Schemes for PDEs**: Part I [KSh] and Part II [KSh2]

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The road-map:

- Analytic presentation $F(x, u, \partial_x^\sigma u) = 0, \Rightarrow$ geometric object $Z_k = \{F = 0\}$.
- Obtain a stable object Z^∞ via the formal theory of integrability. Yields a differential ideal $\mathcal{I}_{Z^\infty}$ from a family of compatible finite-order *algebraic* ideals $\{I_k : k \geq 0\}$.
- Associated canonical algebraic structures - symbolic/characteristic module $\mathcal{M}_\bullet$.
- Geometric integrability conditions on Z^∞ induce algebraic regularity conditions on $\mathcal{M}_\bullet$ (realized via Koszul, Spencer complexes).
- Numerical polynomial $P_{\mathcal{D}}(\mathcal{O}_{Z^\infty})$, via these resolutions of $\mathcal{M}_\bullet$.
- A space $\mathcal{Q}uot_{\mathcal{D}}^P$ parameterizing sub-schemes Z^∞ with $\mathcal{M}_\bullet$ of prescribed regularity, induced by fixing $P \in \mathbb{Q}[t]$ and requiring $P_{\mathcal{D}} = P$.

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A system of N differential equations for a vector-valued function $u = (u^1, \ldots, u^m)$ of variables $x = (x_1, \ldots, x_n)$ is:

$$Z_k := \begin{cases} F_1(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}, \ldots, \frac{\partial^{|\sigma|} u}{\partial x_\sigma}) = 0, \\ \vdots \\ F_N(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}, \ldots, \frac{\partial^{|\sigma|} u}{\partial x_\sigma}) = 0. \end{cases} \tag{1}$$

Derivatives $u^\alpha_\sigma := \frac{\partial^{|\sigma|} u^\alpha}{\partial x_\sigma}$, for $\alpha = 1, \ldots, m$: multi-index notation $\sigma = (\sigma_1, \ldots \sigma_n) \in \mathbb{N}^n$ with *length* $|\sigma| := \sigma_1 + \cdots + \sigma_n \leq k$ (*order*).

Simplifying assumption: The orders of each $F_A, A = 1, \ldots, N$ are the same.

- Z_k is the zero locus of (1) in k-jet space $\underbrace{J^k_X E}_{dim = n + m\binom{n+k}{k}} \ni (x^i, u^\alpha, u^\alpha_\sigma)$ i.e.

$$\mathcal{O}_{Z_k} = \mathcal{O}_{J^k_X E}/I_k, \text{ with } I_k = span\{(1)\}.$$

Remark 1. The structure sheaf $\mathcal{O}_{J^k_X(E)}$ is the k'th-jet sheaf $J^k(E^\vee)$.

Reminder from Algebraic Geometry

First jets can be obtained by considering first order Nilpotent Thickening of Diagonal. Given $\Delta : X \hookrightarrow X \times X$ consider the canonical short exact sequence:

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X\times X} \to \Delta_* \mathcal{O}_X =: \mathcal{O}_\Delta \to 0 \tag{2}$$

Thickening Δ in $X \times X$ we obtain

$$0 \to \mathcal{I}_\Delta^2 \to \mathcal{O}_{X\times X} \to \mathcal{O}_{2\Delta} \to 0 \tag{3}$$

It is then seen that the following s.e.s. holds true:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{2\Delta} \to \mathcal{O}_\Delta \to 0 \tag{4}$$

After projection via $p : X \times X \to X$ we obtain

$$0 \to \Omega_X^1 \to p_*(\mathcal{O}_{2\Delta}) := J^1(\mathcal{O}_X) \to \mathcal{O}_X \to 0 \tag{5}$$

Similarly, we can apply k-fold jet construction to a sheaf $E^\vee$ and obtain:

$$0 \to Sym^k \Omega_X^1 \otimes E^\vee \to J_X^k(E^\vee) \to J_X^{k-1}(E^\vee) \to 0 \tag{6}$$

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The restriction $Sym^k\Omega^1_X \otimes E^\vee\, |_{Z_k}$ is called order k Co-symbol of PDE Z_k.

Differential operators $P : E \to F$ of order $\leq k$ between vector bundles (or coherent sheaves) are equivalently $\mathcal{O}_X$-linear morphisms F_P,

$$P \in \mathcal{D}^{\leq k}_X(E,F) \simeq F_P \in \mathcal{H}om(J^k_X E, F),\ via\ P = F_P \circ j_k.$$

Prolonged operators: $F^{(\ell)}_P : J^{\ell+k}_X E \to J^\ell_X F, \ell \geq 0,\ via\ F^{(\ell)}_P := F_{j_\ell \circ P}$.

- Obtain $Z^\infty = \ker(F^{(\infty)}_P)$, by considering $Z_k := \{F_P = 0\}$ together with all differential consequences

$$Z^\infty := Z_k \cup \bigcup_{\ell \geq 0} Z^{(\ell)}_k \simeq \{D_\sigma F_P = 0 | \sigma \text{ multi-index } |\sigma| \geq 0 (\text{unbounded})\}.$$

- $\mathcal{O}_{Z^{(\ell)}_k} = \mathcal{O}_{J^{k+\ell}_X E}/I_{k+\ell}$, with $I_{k+\ell} = \{F_P, D_\sigma F_P : |\sigma| \leq \ell\}$.

Remark. Ideals $\mathcal{I}_{Z^\infty}$ are 'differentially-generated'= algebraically generated by $D_i(F)$

$$D_i := \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \sum_{|\sigma| \geq 0} u^\alpha_{\sigma+1_i} \frac{\partial}{\partial u^\alpha_\sigma}, \quad (\text{total derivative}). \qquad (7)$$

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What is a PDE?

Forgetting derivatives gives a jet-tower:

$$\cdots \to J^{k+\ell}_X E \xrightarrow{\pi^{k+\ell}_{k+\ell-1}} J^{k+\ell-1}_X E \to \cdots \to J^k_X E \to J^{k-1}_X E \to \cdots \to E \to X.$$

Considering $Z_k = \{F(x, u(x), \partial^\sigma_x u(x) = 0\} \hookrightarrow J^k_X E$, two main operations:

- Projection: The r-fold projection, $\rho_r(Z_k) := \pi^k_{k-r}(Z_k) \subseteq J^{k-r}_X E$.
- Prolongation: $\mathrm{pr}_1(Z_k) := J^1_X(Z_k) \times_{J^1_X(J^k_X E)} J^{k+1}_X E$,

$$\text{Put } p^k := \pi^l|_{Z_k} : Z_k \to X \Rightarrow Z_{k+r} := \mathrm{pr}_r(Z_k) = J^r_X(p_k) \cap J^{q+r}_X E \subseteq J^{q+r}_X.$$

Remark 1. Neither $\rho_r(Z_{k-r})$ nor Z_{k+r} are smooth in general (not relevant for us).

Remark 2. More simply $\rho_k Z_k \subseteq Z_{k-1}$ and $\mathrm{pr}_1(Z_k) \supseteq Z_{k+1}$(not equality!).

Remark 3. They are not inverse operations: take $Z_k \subset J^k_X E$ prolong $Z_{k+r} \subseteq J^{k+r}_X$ and project back to $J^k_X E$, we only get $\rho_r Z_k \subseteq Z_k$ (not equality!).

What is a PDE?

Ex. $Z_0^{wave} := \{F = u_{00} - cu_{ii} = 0\} \hookrightarrow J^2_X E \simeq k[t, x_1, \ldots, x_n, u, u_i, u_{ij}]$.

$$\rightsquigarrow Z^{(\infty)} = \{u_{\sigma 00} = cu_{\sigma ii}, |\sigma| \geq 0\}.$$

Key point (Vinogradov's Philosophy)

A PDE $Z \hookrightarrow J^k_X E$ should be understood in terms of its infinite-prolongation $Z^{(\infty)}$ as a sub-space of $J^\infty_X E$.

Since analytic presentations are just one representative, may lead to ambiguities:

Ex. $Z := \{u_{xx}u_{tt}^2 + u_{tx}^2 + (u_x^2 - u_t)u = 0\}$, and

$$Z' := \begin{cases} u_x = v, \\ u_t = w, \\ v_x w_t^2 + v_t w_x + (v^2 - w)u = 0. \end{cases}$$

Equivalent, but can not be compared at face value (live in different jet-spaces; $Z \subset J^2(E)$ with $\dim(E) = 3$ and $Z' \subset J^1(E')$ with $\dim(E') = 5$.)

What is a PDE?

Thus we can not compare intrinsic structure in a unified way between the two presentations e.g. can not discuss symmetries, solutions, conservation laws etc.

Assemble integrability conditions arising at finite jet/prolongation orders into a single geometric object⇒ Algebraic $\mathcal{D}$-schemes (stacks, higher stacks etc.)

Spencer ($\mathcal{D}$-) Regularity

Infinite prolongations: a system of non-linear PDEs Z of k-th order is a family of sub-schemes $Z^\ell \subset J^\ell_X E$ for all $\ell \in \mathbb{N}$ where $\ell \geq k$. It is given as a tower of surjective submersions with inverse limit $Z^\infty = \varinjlim Z^\ell$.

Prolongation $\Rightarrow$ tower of schemes:

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We start with the differentially stable objects - it is convenient to single out those which arise from the Cartan-Kuranishi prolongation process.

Definition

Let $\mathcal{I}$ be a $\mathcal{D}$-ideal with corresponding closed $\mathcal{D}$-subscheme $Z^\infty \subset J_X^\infty E$. It is *Spencer r-regular* if: (i) $Z_r, \mathrm{pr}_1(Z_r)$ are (possibly singular) schemes and $\mathrm{pr}_1(Z_r) \to Z_r$ is surjective, and (ii) for all $\ell \geq 0$ one has $Z_{\ell+r} = \mathrm{pr}_\ell(Z_r)$. A $\mathcal{D}$-scheme is *Spencer regular* if it is Spencer r-regular for some $r \geq 0$.

Write $\mathrm{Reg}_{\mathcal{D}}(Z) = r$.

* Pictorially:

$$Z \leftarrow Z^1 \leftarrow \cdots \leftarrow Z^{r_0} \twoheadleftarrow Z^{r_0+1} \leftarrow \cdots,$$

highlighting the regular region, defined by all $r \geq r_0$ such that $Z^{r+1} = \mathrm{pr}_1(Z^r)$. Everywhere else, we ask that $Z_{r+1} \subseteq \mathrm{pr}_1(Z_r)$, for all $r \geq k$.

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Intuition from Algebraic Geometry

Equation ideal $\mathcal{I}$ is stable under an action by $\mathcal{D}_X$ (a D-ideal) and $\mathcal{O}_{Z^\infty}$ is an algebra (also stable by $\mathcal{D}$) e.g. $\mathcal{O}_{Z^\infty} \in \mathrm{CAlg}_X(\mathcal{D}_X)$.

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Concept	**Algebraic Geometry**	**$\mathcal{D}$-Geometry**
Formula	$P(x) = 0$	$F(x, u, \partial^\sigma u) = 0$
Algebraic structure	Commutative $\mathbf{k}$-algebra A	Commutative $\mathcal{D}$-algebra $\mathcal{A}$
Free structure	$P \in \mathbf{k}[x]$	$F \in \mathcal{A} := \mathcal{O}_{J_X^\infty E}$
Solution space	$\{x \in A \mid P(x) = 0\}$	$\{a \in \mathcal{A} \mid F(a) = 0\}$
Affine object	$\mathrm{Spec}_k(A)$	$\mathrm{Spec}_{\mathcal{D}}(\mathcal{A})$
Representability	$\mathrm{Sol}_k(P = 0) \simeq \mathrm{Spec}(A/P)$	$\mathrm{Sol}_{\mathcal{D}}(F = 0) \simeq \mathrm{Spec}_{\mathcal{D}}(\mathcal{O}_{J_X^\infty E}/\mathcal{I}_{Z^\infty})$

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Is there a moduli space of $\mathcal{D}$-subschemes?

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Hilbert and Quot Schemes in Algebraic Geometry

All schemes are locally Noetherian over an algebraically closed field k.

X/S is projective and $\mathcal{O}(1)$ a relatively ample line bundle. P is a fixed numerical polynomial.

Recall: Hilbert functor $\underline{Hilb}^P(X/S) : Sch_{/S}^{op} \to Sets$, associates to an S-scheme T, the set

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Theorem (Grothendieck)

The Hilbert functor is representable by a projective scheme.

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Fact: By restricting to structure sheaves of subschemes with a fixed P, we can find an integer m depending only on P that works simultaneously for the structure sheaf of *every* subscheme with Hilbert polynomial equal to P (uniform bound).

Definition

A coherent sheaf $F \in Coh(X)$ is *Castelnuovo-Mumford m-regular* if $H^i(X, F(m-i)) = 0, i > 0$.

This leads to an embedding of the Quot scheme into a suitable Grassmann variety.

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$$\delta(w_1 \cdots w_p \otimes v) = \sum_{i=1}^{p} w_1 \cdots w_{i-1} w_i \cdots w_p \otimes w_i \wedge v.$$

Remark. $\partial^2 = 0, \delta^2 = 0$, and $\mathcal{R}_\bullet(V)^\vee \simeq K_\bullet(V^\vee)$.

For a module M, over a polynomial ring, tensoring the Koszul sequence gives $K_\bullet(M) \simeq K_\bullet(V) \otimes M$.

If M is additionally graded, the Koszul complex is *bi-graded*:

$$H_{p,q}(K_\bullet(M)) := H\big(\wedge^{p+1} V \otimes M_{q-1} \to \wedge^{p} V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1}\big).$$

The notion of regularity uses these spaces: M is *r-regular* if: $H_{p,q}\big(K_\bullet(M)\big) = 0, \forall p, q \geq r$.

Definition

The *regularity* of M is $\mathrm{reg}_{CM}(M) := \inf_r \{H_{p,q}\big(K_\bullet(M)\big) = 0, \forall p, q \geq r\}$.

Algebraic regularity

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Algebraic $\mathcal{D}$-Geometry of Non-linear PDEs

The 'Guiding Idea'

Geometric theory of PDEs is similar to ordinary algebraic geometry.

Algebraic Geometry: $P_1(x_1,\ldots,x_N)=0,\ldots,P_N(x_1,\ldots,x_n)=0 \Rightarrow$ a geometric object i.e. *schemes over k-algebras*:

$$\underset{e.g. Hilb_X^P, Quot^P}{\mathrm{Sch}_k \subset \mathrm{Fun}(\mathrm{CALG}_k^{op}, \mathrm{SETS})} \rightsquigarrow S : \begin{cases} P_1(x_1,\ldots,x_n)=0 \\ \vdots \\ P_N(x_1,\ldots,x_n)=0 \end{cases}.$$

$\mathcal{D}$-Geometry: $F_1(x,\ldots,u_{[\sigma]},\ldots)=0,\ldots,F_N(x,\ldots,u_{[\sigma]},\ldots)=0 \Rightarrow$ a geometric object i.e. *schemes over $\mathcal{D}$-algebras*:

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Basic idea

Define a *set*,

$$[\mathcal{I}_X] \in \mathcal{Q}_{\mathcal{D}_X}(n,m,N,k) \;\rightsquigarrow\; \begin{array}{ccccc} \mathcal{I}_{Z^\infty} & \longrightarrow & \mathcal{O}_{J^\infty_X E} & \overset{q}{\twoheadrightarrow} & \mathcal{O}_{Z^\infty} \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \beta} \\ \mathcal{I}_{Z'_\infty} & \longrightarrow & \mathcal{O}_{J^\infty_X E} & \overset{q'}{\twoheadrightarrow} & \mathcal{O}_{Z'_\infty}, \end{array}$$

where α, β are isomorphisms of $\mathcal{D}_X$-algebras.

Impose that the system $\{F_A = 0 : A = 1, \dots, N\}$, or equivalently the ideal $\mathcal{I}_{Z^\infty}$ satisfies

$$\underbrace{\mathrm{Sol}_{\mathcal{D}}(\mathcal{I}_{Z^\infty}) \neq \mathrm{Spec}_{\mathcal{D}}(\mathcal{O}_{J^\infty_X E})}_{\text{Non-trivial}}, \quad \textit{and} \quad \underbrace{\mathrm{Sol}_{\mathcal{D}}(\mathcal{I}_{Z^\infty}) \neq \emptyset}_{\text{Consistent}}. \tag{8}$$

Formal integrability is assumed - the ideals contain all possible integrability conditions.

Our original question

Question

Is the space $\mathcal{Q}_{\mathcal{D}_X}$ of $\mathcal{D}$-ideal sheaves representable?

What do we need?

- Identify numerical characterizations to parameterize our objects;
- Obtain 'regularity' compatible with cohomological vanishing
- Show uniform boundedness in flat families

Question

Is there a derived enhancement $\mathbb{R}\mathcal{Q}_{\mathcal{D}_X}$?

Algebraic $\mathcal{D}$-Geometry $\rightsquigarrow$ Derived $\mathcal{D}$-Geometry

$$\mathrm{Sch}_X(\mathcal{D}_X) \subset \mathrm{Fun}(\mathrm{CAlg}_X(\mathcal{D}_X)^{op}, \textsc{Sets}) \rightsquigarrow \mathbf{dStk}_X(\mathcal{D}_X) \subset \mathbf{dPStk}_X(\mathcal{D}_X).$$

Geometric symbols of (non-linear) PDEs admit a purely algebraic description.

Let V, W be finite-dimensional vector spaces, $dimV = n$.

Consider $Sym^k(V^\vee) \otimes W, k \geq 0$.

Let $\mathcal{N}_k \subseteq Sym^k(V^\vee) \otimes W$ be a vector subspace.

First-prolongation is $\mathrm{pr}_1(\mathcal{N}_k) := \{\mathcal{P} \in Sym^{k+1}(V^\vee) \otimes W | \delta(\mathcal{P}) \in \mathcal{N}_k \otimes V^\vee\}$.

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Definition

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Remark: $\mathrm{pr}_{r+1}(\mathcal{N}_k) := \mathrm{pr}_1(\mathrm{pr}_r(\mathcal{N}_k)), r \in \mathbb{N}$ or $\mathrm{pr}_r(\mathcal{N}_k) = (\otimes_{i=1}^r V^\vee \otimes \mathcal{N}_k) \cap (Sym^{q+r}(V^\vee) \otimes E)$.

We can construct a graded module $\mathcal{M}_\bullet := \oplus_k \mathcal{N}_k \Rightarrow K_\bullet(\mathcal{M}_\bullet) \Rightarrow H_{p,q}\big(K_\bullet(\mathcal{M}_\bullet)\big)$.

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Roughly

An involutive non-linear PDE is one whose symbol module is regular.

The (geometric) symbol of an involutive non-linear PDE induces at each point a polynomial (co)module - characteristic module.

Approach: Use Koszul homology or $\mathbb{R}$-dual non-linear Spencer cohomology to determine the length of finite resolutions (determine the regularity, analogous to Castelnuovo-Mumford).

- Only looking at involutive PDEs is not a huge restriction - analytic systems can always be made involutive (Cartan-Kuranishi Theorem).
- Most examples in Mathematical Physics are of this type e.g. Yang-Mills, Einstein, Einstein-Maxwell, Higher p-form theories...

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Compatibility with 'cohomological vanishing'

Involutivity and the degree of involution correspond to a certain cohomological vanishing criteria (Cartan, Goldschmidt and Spencer)

Spencer cohomology $\sim$ Sheaf cohomology for PDEs (the sheaf is the solution sheaf).

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Spencer Cohomology I

Observation: for each $k \geq 1$, we have

$$0 \to \underbrace{Sym^k(\Omega_X^1) \otimes E}_{\text{(co)symbols live here}} \to J_X^k E \xrightarrow{\pi_{k-1}^k} J_X^{k-1} E \to 0.$$

The isomorphism is:

$$Sym^k(\Omega_X^1) \otimes E \xrightarrow{\simeq} Ker(J_X^k E \to J_X^{k-1} E), \tag{9}$$

defined by $df_1 \cdots df_k \otimes e \mapsto \delta_{f_1,\cdots,f_k}(j_k)(e), e \in E$, where $\delta_{f_1,\cdots,f_k} := \delta_{f_1} \circ \cdots \delta_{f_k}$, is the nested commutator, where $\delta_{f_i} := [f_i, -]$.

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defined by $df_1 \cdots df_k \otimes e \mapsto \delta_{f_1,\cdots,f_k}(j_k)(e), e \in E$, where $\delta_{f_1,\cdots,f_k} := \delta_{f_1} \circ \cdots \delta_{f_k}$, is the nested commutator, where $\delta_{f_i} := [f_i, -]$.

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Spencer Cohomology I

Observation: for each $k \geq 1$, we have

$$0 \to \underbrace{Sym^k(\Omega^1_X) \otimes E}_{\text{(co)symbols live here}} \to J^k_X E \xrightarrow{\pi^k_{k-1}} J^{k-1}_X E \to 0.$$

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Spencer cohomology

Given a coherent sheaf E on X, the k-th Jet-Spencer complex of E is,

$$J^k Sp^\bullet_X(E) := 0 \to E \xrightarrow{j_k} J^k_X E \xrightarrow{S} \Omega^1_X \otimes_{\mathcal{O}_X} J^{k-1}_X E \xrightarrow{S} \cdots, \qquad (10)$$

with $S(\omega \otimes j_k(e)) := d\omega \otimes j_{k-1}(e)$. Projection $J^k_X E \to J^{k-1}_X E$ induces a morphism of complexes:

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For each $k \geq 1$ kernel of Jet-Spencer complexes:

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It is a complex of $\mathcal{O}_X$-modules with differential $\delta : (\omega \otimes u \otimes e) \mapsto (-1)^{|\omega|} \omega \wedge i(u) \otimes e$, with $i : S^{k-r}\Omega^1_X \hookrightarrow \Omega^1_X \otimes S^{k-r-1}\Omega^1_X$.

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In local charts

Restricting to Z_k, we obtain δ-Spencer complexes for cosymbols $\mathcal{N}_k$.

Consider $\iota_k : Z_k = \{F(x, u, u_{[\sigma]}) = 0\} \hookrightarrow J^k_X(E)$:

$$T_\theta J^k_X E = \text{span}\Big\{v := \dot{x}^i \frac{\partial}{\partial x_i} + \dot{u}^\alpha_\sigma \frac{\partial}{\partial u^\alpha_\sigma}\Big\}, \text{ and } T_\theta Z_k = \{v \in T_\theta J^k_X E | TF(v) = 0\}.$$

A linear system for the coefficients of v:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x^i}(\theta) \cdot x^i + \sum_{1 \leq \alpha \leq m, 1 \leq \sigma \leq \text{ord}(F)} \frac{\partial F}{\partial u^\alpha_\sigma}(\theta) \dot{u}^\alpha_\sigma = 0.$$

The geometric symbol $\mathcal{N}_k$, of Z_k is the family of vector spaces:

$$(\mathcal{N}_k)_z := T_z Z_k \cap T_{J^k_X E / J^{k-1}_X E, z} \simeq T_{\pi^k_{k-1}|Z_k}, \text{ (relative tangents).}$$

Coordinates

For $Z_k = \{F = 0\}$, then $\mathcal{N}_k = \sum_{1 \leq \alpha \leq m, |\sigma| = \text{ord}(F)} \frac{\partial F}{\partial u^\alpha_\sigma}(\theta) \dot{u}^\alpha_\sigma = 0$.

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Symbols and Characteristic Modules, Algebro-geometrically

Consider relative differentials of dg of holomorphic functions $g \in \mathcal{O}_{J^\ell_X E}$ i.e. $d(g) \in \Omega^1_{J^\ell_X E / J^{\ell-1}_X E} \simeq \mathcal{O}_{J^\ell_X E}[\xi_1, \ldots, \xi_n]_k$, with $(x;\xi) \in T^*X$.

Put $\mathcal{O}_{Z_r} = \mathcal{O}_{J^r_X E}/I_r$.

Order r symbol: $\mathcal{N}_r \subset \oplus_\alpha \mathcal{O}_{Z_r}[\xi_1, \ldots, \xi_n]_r d(u^\alpha)$, generated by classes dF mod I_r of $F \in I_r$.

Symbol: the associated graded module generated by $\mathcal{N}_r$.

Characteristic module: graded quotient $\mathcal{M} := \bigoplus_r \mathcal{O}_{Z_r}[\xi_1, \ldots, \xi_n]/\mathcal{N}_r$.

$*$ Alternatively, look at conormal I_r/I_r^2 of Z_r, as sub-sheaf of $\Omega^1_{J^r_X E} \otimes \mathcal{O}_{Z_r}$. Then,

$$\mathcal{M}_r \simeq \Omega^1_{Z_r/Z_{r-1}}.$$

From $Z^\infty \Rightarrow F^r Z := Z_r \subset J^r_X(E)$,

$$\mathcal{O}_{F^r Z} \otimes_{\mathcal{O}_{F^{r-1}Z}} \Omega^1_{F^{r-1}Z/X} \rightarrow \Omega^1_{F^r Z/X} \rightarrow \Omega^1_{F^r Z/F^{r-1}Z} \rightarrow 0. \tag{11}$$

Applying $\mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}(F^r Z)} (-)$:

$$\mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^{r-1}Z}} \Omega^1_{F^{r-1}Z/X} \rightarrow \mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^r Z}} \Omega^1_{F^r Z/X} \rightarrow \mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^r Z}} \Omega^1_{F^r Z/F^{r-1}Z} \rightarrow 0.$$

In more generality:

Consider a family of prolonged operators: $\{F_P^{(\ell)} : J_X^{k+\ell}E \to J_X^{\ell}F\}_{\ell\geq 0}$.
There is an induced family of (prolonged) symbol maps:

$$\sigma_{\ell,k}(F_P) : \mathcal{S}ym^{k+\ell}_{\mathcal{O}_X}(\Omega^1_X) \otimes_{\mathcal{O}_X} E \to \mathcal{S}ym^{\ell}_{\mathcal{O}_X}(\Omega^1_X) \otimes F. \tag{12}$$

A family of symbolic modules: $\{\mathcal{N}^{k+\ell} := \ker\big(\sigma_{\ell,k}(F_P)\big)\}_{\ell\geq 0}$

The Spencer δ-complex:

$$0 \to \mathcal{N}^{k+\ell} \xrightarrow{\delta} \Omega^1_X \otimes \mathcal{N}^{k+\ell-1} \to \Omega^2_X \otimes \mathcal{N}^{k+\ell-2} \to \cdots$$

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Summary

The situation is summarize by the commuting diagram,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 & \\
 & & \downarrow & & \downarrow & & & & \downarrow & \\
0 & \longrightarrow & \mathcal{N}^{(r)} & \longrightarrow & \mathcal{N}^{(r-1)} \otimes \Omega_X^1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{N}^{(r-n)} \otimes \Omega_X^n & \to 0 \\
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0 & \longrightarrow & \mathrm{pr}_r Z & \longrightarrow & \mathrm{pr}_{r-1} Z \otimes \Omega_X^1 & \longrightarrow & \cdots & \longrightarrow & \mathrm{pr}_{r-n} Z \otimes \Omega_X^n & \to 0 \\
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Degree of Involution

Interpretation! $\mathcal{D}$-involutivity $\sim$ Serre vanishing (for sheaf of solutions).

Definition: Let Z be formally integrable and involutive. Its *degree of involution* is

$$\mathrm{reg}_{Sp}(\mathcal{I}) := \inf_{q_0 \in \mathbb{N}} \left\{ H_\delta^{p,q}(\mathcal{M}_\bullet) = 0, \text{ for all } 0 \leq p \leq \dim X, q \geq q_0 \right\}.$$

Proposition/Observation $(\star)$

Suppose that $\mathcal{I}$ is formally integrable and Spencer regular. Then its degree of involution, agrees with the Spencer-regularity, and thus agrees with the Castelnuovo-Mumford regularity of its (linearized) symbolic $\mathcal{D}$-module.

Proof. $\mathcal{I}_{Z^\infty} = \{F_A\} \rightsquigarrow \sigma(\mathcal{I}_{Z^\infty}) := \{\sigma_{\ell,k}(\ell_{F_A})\}$. They are homogeneous polynomials determining a 'symbol ideal' in a polynomial algebra (on T^*X). Spencer sequences and Koszul sequences are dual.

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Summary: The role of involution

Pure geometric reasoning will not suffice to known when new integrability conditions stop occurring.

Use also algebraic theory: formal integrability is often not sufficient and one introduces ‘involution’.

Rough idea: A natural polynomial structure lies hidden in the inner geometry of the jet bundle formalism.

This allows us to associate with any $\{F = 0\}$ a $\mathcal{D}$-ideal $\mathcal{I}$ and graded symbol module $\mathcal{M}_\bullet$ over a polynomial algebra.

Constructing integrability conditions of $\{F = 0\}$ is then formalised via the syzygies of $\mathcal{M}$.

Remark: We make contact with classical questions in commutative algebra and algebraic geometry.

Summary: The role of involution

Pure geometric reasoning will not suffice to known when new integrability conditions stop occurring.

Use also algebraic theory: formal integrability is often not sufficient and one introduces ‘involution’.

Rough idea: A natural polynomial structure lies hidden in the inner geometry of the jet bundle formalism.

This allows us to associate with any $\{F = 0\}$ a $\mathcal{D}$-ideal $\mathcal{I}$ and graded symbol module $\mathcal{M}_\bullet$ over a polynomial algebra.

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Numerical polynomials

By Prop. ($\star$): associate a $P_{\mathcal{D}}(\mathcal{O}_{Z^\infty})$ using resolutions of symbol/characteristic module $\mathcal{M}_\bullet$. Such a resolution exists, expressible via Spencer δ-cohomology.

Encodes the dimensions of $\mathrm{Sol}(\mathcal{I}_{Z^\infty})$ (Cartan characters) and may be determined geometrically from the dimension of the support the sheaf $\mathcal{M}_\bullet$.

Define characteristic varieties $\mathrm{Char}_{\mathcal{D}}(Z^\infty) := supp(\mathcal{M}_\bullet) \subset Z^\infty \times_X T^*X$

For each point $z_\infty \in Z^\infty$, corresponding to solution, pull-back $\mathcal{M}_\bullet$ is $\mathcal{D}$-module and Char is usual characteristic variety $\subset T_X^*$.

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$$P_{\mathcal{D}}(\mathcal{O}_{Z^\infty}, \ell) = \sum_{p,q} (-1)^{p+q} \dim \mathcal{H}_{Sp}^{p,q}(\mathcal{M}_\bullet) \cdot \binom{\ell + n - p - q - 1}{n-1}. \qquad (14)$$

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Interpretations of Spencer cohomology: $H^{p,q}(\mathcal{M}_\bullet)$ encode e.g. $H^{*,1}(\mathcal{M}_\bullet)$ count $\sharp$ generators of the symbol module, $H^{p,1}(\mathcal{M}_\bullet)$ count $\sharp$ equations of order p, and $H^{*,2}(\mathcal{M}_\bullet)$ counts compatibility conditions.

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Summary by analogy

Comparing with Algebraic Geometry $\Rightarrow$ numerical classification of $\mathcal{D}$-finitely generated ideals associated with differentially consistent systems

Concept	**Algebraic Geometry**	**$\mathcal{D}$-Geometry**
Sheaf	Algebraic ideal $\mathcal{I}_Z \subset \mathcal{O}_X$	$\mathcal{D}$-ideal $\mathcal{I}_{Z^\infty} \subset \mathcal{A}^\ell$
Geometric	Sub-scheme Z	$\mathcal{D}$-subscheme $Z^\infty \hookrightarrow J_X^\infty E$
Numerical	Hilbert polynomial $P_{\mathcal{O}_Z}$	$\mathcal{D}$-Hilbert polynomial $P_{\mathcal{O}_{J_X^\infty E}/\mathcal{I}_{Z^\infty}}$
Graded ring	$R_* = \oplus_t \Gamma(X, \mathcal{O}_X(t))$	$\mathcal{O}_{T^*(X;Z^\infty)}(*) = \oplus_t \mathcal{O}_{T^*(X;Z^\infty)}(t)$
Sheaves	$\mathcal{F} \in \mathrm{Coh}(X)$	$\Omega^1_{Z^\infty} \in \mathrm{Mod}(\mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$
Graded module	$M_* \in \mathrm{Mod}^{gr}(R_*)$	$\mathrm{Gr}(\Omega^1_{Z^\infty}) \in \mathrm{Mod}^{gr}(\mathcal{O}_{T^*(X;Z^\infty)})$
Sheaf theoretic	Support	Microsupport (characteristic variety)
Boundedness	Castelnuovo-Mumford regularity	Degree of involutivity
Moduli	Hilbert scheme $Hilb_X$	$\mathcal{D}$-Hilbert scheme $\mathbf{Hilb}_{\mathcal{D}}^P$

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The $\mathcal{D}$-Hilbert Functor

Recap so far

Introduced parameterizing space of isomorphism classes of exact sequences,

$$\mathcal{Q}uot(n,m,N,k) := \frac{\left\{\text{Sequences } (\mathcal{I}_{Z^\infty} \to \mathcal{O}_{J^\infty E} \to \mathcal{O}_{Z^\infty})\right\}}{\text{Isomorphism}} \tag{15}$$

Restrict:

$$\mathcal{Q}uot^{\text{inv}}(n,m,N,k) := \left\{[\mathcal{I}_{Z^\infty}] \in (15) \,|\, \text{(Spencer-regular)}, \text{ integrable } \right\}. \tag{16}$$

Assigned numerical characterizations $P_{\mathcal{D}}(\mathcal{O}_{Z^\infty}) \in \mathbb{Q}[z]$, tracking homological complexity of symbolic system.

Goal

Prove (16) defines a moduli functor.

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Main Results of [KSh]

The main result constructs the moduli functor and proves its representability.

Theorem (K-S, arXiv:2507.07937 (2025))

Let X be a smooth D-affine algebraic variety of dimension n. Let $m \in \mathbb{N}$ represent the number of dependent variables and fix $P \in \mathbb{Q}[t]$. Then there exists a moduli functor classifying formally integrable (non-singular) algebraic differential systems with specified (Spencer) regular symbolic behavior whose numerical polynomial $P_{\mathcal{D}}$ is equal to P :

$$\underline{\mathcal{Q}uot}_{\mathcal{D}_X}(P; n, m)^{\text{inv}} : \mathrm{Sch}_X(\mathcal{D}_X)^{op} \to \mathrm{Sets},$$

There exists a sub-functor consisting of Spencer semi-stable differential ideals representable by an (ind)-finite-type $\mathcal{D}$-scheme.

The notion of $\mathcal{D}$-regularity and of the degree of involution behave appropriately with respect to sequences.

Proposition (K-S, arXiv:2507.07937 (2025))

If Z is Spencer m-regular, then it is Spencer m'-regular for all $m' \geq m$. Moreover, suppose there are smooth morphisms of $\mathcal{D}$-schemes $Z' \to Z \to Z''$, with $\mathcal{D}$-ideals $\mathcal{I}', \mathcal{I}, \mathcal{I}''$, respectively, inducing a short exact sequence of symbols. Suppose each $\mathcal{D}$-scheme is Spencer regular. The following statements hold:

- If Z' and Z'' are m-regular, then so is Z;
- If Z' is $(m+1)$-regular and Z is m-regular then Z'' is $(m+1)$-regular;
- If Z is m-regular and Z'' is $(m-1)$-regular then Z' is m-regular.

Method of proof (á la Grothendieck/Douady)

- Analog of Mumford's boundedness criterion:

Proposition (K-S, arXiv:2507.07937 (2025))

Fix a numerical polynomial P. Consider a family $\{Z_i^{\infty}\}_{i\in I}$ of Spencer-regular algebraic PDEs. The following two statements are equivalent:

- The family is bounded;
- The set of $\mathcal{D}$ Hilbert polynomials $\{P_{\mathcal{D}}(\mathcal{O}_{Z_i^{\infty}})\}$ is finite and there is a uniform bound on the involutivity degrees i.e. $\mathrm{Reg}_{\mathcal{D}_X}(Z_i^{\infty}) \leq \rho, \forall i \in I$.

Namely, fixing an ambient $\mathcal{D}$-scheme Z (e.g. $J_X^{\infty}E$) and P, there exists some integer $\rho = \rho(P)$ such that for $Z_i^{\infty} \subset Z$ with $\mathcal{D}$-Hilbert polynomial $P_i = P$, its $\mathcal{D}$-ideal sheaf has $\mathcal{D}$-regularity ρ.

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Non-characteristic restriction

- PDE Leftschetz hyperplane theorem:

Theorem (K-S, arXiv:2507.07937 (2025))

Let $\mathcal{I}_{Z^\infty}$ be a $\mathcal{D}$-involutive $\mathcal{D}$-ideal sheaf. For $j : H \subset T^*X$ a non-microcharacteristic analytic subset, denote by $\mathcal{I}_H$ the $\mathcal{D}$-ideal obtain by non-characteristic restriction. Then $\mathrm{Reg}_{\mathcal{D}}(\mathcal{I}_X) = m$ if and only if $\mathrm{Reg}_{\mathcal{D}}(\mathcal{I}_H) = m$.

Spencer regularity is thus preserved under restriction to non-characteristic subvarieties. This allows us to control local-to-global properties of symbolic data.

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A second result is the calculation of the Zariski tangent $\mathcal{D}$-module.

Theorem (Theorem (K-S, arXiv:2507.07937 (2025)))

The tangent space at a point $[\mathcal{I}]$ controlling first-order deformations of $\mathcal{I}$ as a $\mathcal{D}$-ideal, is given by

$$T_{[\mathcal{I}]}\mathbf{Hilb}_{\mathcal{D}_X}(J_X^{\infty}E) \simeq Hom_{\mathcal{D}_X}(\mathcal{I}, \mathcal{O}(J_X^{\infty}E)/\mathcal{I}).$$

There is a space of obstructions $\mathcal{O}bs_{[\mathcal{I}]}$ which is a filtered $\mathcal{D}$-module over the $\mathcal{D}$-scheme defined by $\mathcal{I}$ whose associated graded algebra is isomorphic to a sub-quotient of the Ext^1 group giving the (truncated) Spencer cohomology determined by the symbol of $\mathcal{I}$.

GIT Quotients and Moduli of Spencer stable modules

One can prove a moduli space of Spencer stable finitely presented modules (PDE analog of moduli of coherent sheaves).

There exists a locus: $\mathcal{Q}uot_{\mathcal{D}}^{Sp-ss,inv} \subset \mathcal{Q}uot_{\mathcal{D}}^{inv}$, and for algebraic pseudogroups G acting on points of $\mathcal{Q}uot_{\mathcal{D}}^{inv}(P,n,m)$, we describe the invariant quotient:

$$\mathcal{Q}uot_{\mathcal{D}}^{Sp-ss,inv}(P,n,m)//\mathcal{G}.$$

With $\mathcal{G}$ an Algebraic Lie pseudogroup: a family of algebraic sub-groups $\{G^k\}$ of k-jets of bundle isomorphisms ϕ acting on E (covering diffeomorphisms $\underline{\phi}$ of X).

Actions lift to k-jet spaces: $\phi^{(k)} : J_X^k E \to J_X^k E, [s]_x^k \mapsto [\phi \circ s \circ \underline{\phi}^{-1}]_{\underline{\phi}(x)}^k$, functorially in E, X and jet-order, and restrict to PDEs $Z_k \subset J_X^k E$:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow{\underline{\phi}} & Y \end{array} \quad \Rightarrow \quad \begin{array}{ccc} J_X^k E & \xrightarrow{\phi^{(k)}} & J_Y^k F \\ \downarrow & & \downarrow \\ J_X^\ell E & \xrightarrow{\phi^{(\ell)}} & J_Y^\ell F \end{array}$$

We care about $X = Y$.

We describe the moduli space as a type of PDE-theoretic GIT-quotient.

- $\mathcal{D}$-geometric Luna slice theorem:

Theorem (K-S, arXiv:2507.07937 (2025))

Given an algebraic action by a (formally integrable) Lie pseudogroup $G := \{G^k\}_{k\geq 0}$, on a $\mathcal{D}$-scheme with ideal $\mathcal{I}$, the quotient equation exists and is again an algebraic $\mathcal{D}$-scheme, with ideal $\mathcal{I}^G$. In particular, the (categorical) quotient $\mathcal{D}$-scheme $[Z/G] := \{[Z^k/G^k]\}_{k\geq 0}$, corresponds to a geometric quotient.

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Application of Spencer-stability to Non-abelian Hodge and DUY theorems

Solutions to Differential Geometric extremal problems correspond to Algebro-Geometric stability questions.

Theorem (Donaldson '85, Uhlenbeck-Yau '85)

Let (X, ω) be a compact Kähler manifold. A holomorphic vector bundle $E \to (X, \omega)$ admits a Hermitian-Einstein metric if and only if it is slope polystable.

Interpretation: *Existence of PDE solutions* $\leftrightsquigarrow$ *Stability.*

Automorphisms: Stability $\Rightarrow \mathrm{End}(E) = \mathbb{C} \cdot \mathrm{id}$.

Remark

Many challenging and interesting open conjectures (e.g. Thomas-Yau, Yau-Tian-Donaldson, higher rank cases, hypercritical phases, relations to HMS).

T. Bridgeland, *Ann. of Math.* 166 (2007) 317–345

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T. Bridgeland, *Ann. of Math.* 166 (2007) 317–345

Solutions to Differential Geometric extremal problems correspond to Algebro-Geometric stability questions.

Theorem (Donaldson '85, Uhlenbeck-Yau '85)

Let (X, ω) be a compact Kähler manifold. A holomorphic vector bundle $E \to (X, \omega)$ admits a Hermitian-Einstein metric if and only if it is slope polystable.

Interpretation: *Existence of PDE solutions* $\leftrightsquigarrow$ *Stability.*

Automorphisms: Stability $\Rightarrow \mathrm{End}(E) = \mathbb{C} \cdot \mathrm{id}$.

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Relations to stability

The connection to stability questions we focus on arises from the *Non-Abelian Hodge Correspondence*.

Theorem (Corlette, Simpson)

Let X be smooth projective and let (V, ∇) be a complex local system with reductive monodromy. Let (E, θ) be the associated polystable Higgs bundle. Then

$$H^*(X, (V, \nabla)) \cong H^*(X, (E, \theta)).$$

Recall: A *Higgs bundle* (E, θ) of rank m is $\mathcal{O}_X(1)$-stable (resp. semistable) if for all θ invariant subsheaves $F \subset E$,

$$p(F) < p(E), \quad \big(\text{resp. } p(F) \leq p(E)\big),$$

where $p(F) := \chi\big(X; F \otimes \mathcal{O}_X(m)\big)/\mathrm{rk}(F)$, is the reduced Hilbert polynomial of F.

Remark. The Spencer-type polynomial stability encodes and generalizes well-known sheaf-theoretic stability conditions.

Moduli Space	Stability / Boundedness Condition
Moduli of coherent sheaves	Gieseker/slope stability
Hilbert/Quot moduli functors	Castelnuovo–Mumford regularity
Moduli of relative local systems	Semisimplicity / reductivity
Moduli of Higgs bundles	Slope (poly)stability
$\mathcal{D}$**-Hilbert functor**	*Spencer stability*

Works for non-linear PDEs on vector bundles, coherent sheaves, complexes, objects $\mathrm{D}^b(X)$, *objects with stacky/derived structure etc.*

Observation

Spencer-stability of PDEs imposed on *bundles* $\xrightarrow{\text{Formal solutions}}$ Sheaf-stability of bundles/sheaves.

Application: Equation of flat connections

Let (X, ω) be a compact Kähler manifold and (E, ∇) a holomorphic vector bundle of rank m with a flat connection $\nabla : E \to E \otimes \Omega^1_X$. Consider the Atiyah algebroid of E:

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{At}(E) \longrightarrow T_X \longrightarrow 0,$$

where $\mathrm{At}(E)$ consists of first-order differential operators on E with symbol in T_X. Dually, consider $J^1_X E$ the first jet bundle of E, and $J^\infty_E J^1_X E$ its infinite jet bundle. The flatness of ∇ is encoded by a differential ideal

$$\mathcal{I}_\nabla \subset \mathcal{A} := \mathcal{O}(J^\infty_E J^1_X E)$$

generated by the differential equations defining the curvature. Denote

$$\mathcal{B}_\nabla := \mathcal{A}/\mathcal{I}_\nabla,$$

the $\mathcal{D}_X$-algebra of functions. The tangent space to the $\mathcal{D}$-Hilbert functor at $[\mathcal{I}_\nabla]$ is

$$T_{[\mathcal{I}_\nabla]}\mathrm{Hilb}_{\mathcal{D}}(\mathcal{A}) \simeq \mathrm{Hom}_{\mathcal{A}[\mathcal{D}_X]}(\mathcal{I}_\nabla, \mathcal{B}_\nabla).$$

Consider the associated graded of the jet filtration on $\mathcal{I}_\nabla$:

$$\mathrm{gr}(\mathcal{I}_\nabla) = \bigoplus_{k\geq 0} F^k\mathcal{I}_\nabla / F^{k+1}\mathcal{I}_\nabla,$$

where $F^k\mathcal{I}_\nabla$ is the k-th jet order.

Then, the Spencer cohomology computes

$$\mathcal{H}^{p,q}_{\mathrm{Sp}}(\mathrm{gr}(\mathcal{I}_\nabla)) \simeq H^q(X, \Omega^p_X \otimes \mathrm{End}(E)).$$

There is a *stability-type* correspondence providing a purely PDE analog of DUY and NAH theorems.

Theorem ((K-S, 2025)

Let (X, ω) be a compact Kähler manifold. Then a holomorphic vector bundle E admits a Hermitian-Yang-Mills metric if and only if the associated flat connection ∇ defines a Spencer-polystable differential ideal $\mathcal{I}_{(E,\nabla)}$.

The canonical ideal $\mathcal{I}_{(E,\nabla)}$, corresponds to the equation of flat connections.

It encodes the infinitesimal geometry of flat bundles.

Further details

Connections on E are identified with sections of the affine bundle $\pi_{1,0} : J^1(E) \longrightarrow E$. Put $D(\Omega^p_E)$ the module of Ω^p_E-valued derivations.

Connection form and Frölicher–Nijenhuis bracket

A section ∇ of this bundle determines a decomposition $d = \bar{U}_\nabla + U_\nabla$, where $\bar{U}_\nabla \in D(\Omega^1_E)$ is defined by

$$X\left(\bar{U}_\nabla(f)\right) := (\nabla X_M)(f), \qquad X \in D(E),\ f \in C^\infty(E),$$

and $U_\nabla := d - \bar{U}_\nabla$ is the vertical connection form.

$$[\![\cdot,\cdot]\!] : D(\Omega^p_E) \times D(\Omega^q_E) \longrightarrow D(\Omega^{p+q}_E)$$

extends the Lie bracket of vector fields: ∇ *flat* $\Longleftrightarrow [\![\bar{U}_\nabla, \bar{U}_\nabla]\!] = 0.$

Intrinsic definition

The *equation of flat connections* is the submanifold

$$Z_{\text{flat}} := \left\{ \theta_1 \in J^1(E) \;\middle|\; [\![\bar{U}_\nabla, \bar{U}_\nabla]\!]_{\pi_{1,0}(\theta_1)} = 0 \right\},$$

where $\theta_1 = [\nabla]^1_\theta$ is the 1-jet of any local representative connection ∇.

- The value of $[\![\bar{U}_\nabla, \bar{U}_\nabla]\!]$ at θ depends *only* on $\theta_1 \in J^1(E)$, *not* on the choice of ∇.

$$\left\{ \begin{array}{c} \text{Spencer-polystable differential ideals} \\ \mathcal{J} \subsetneq \mathcal{I}^{\nabla} \\ P_{\mathcal{D}}(\mathcal{O}_{\mathcal{S}}) \leq P_{\mathcal{D}}(Z_{\nabla}) \end{array} \right\} \longleftrightarrow$$

$$\left\{ \begin{array}{c} \theta\text{-invariant polystable subbundles} \\ F \subsetneq (E, \theta) \\ p(F) \leq p(E) \end{array} \right\}.$$

Example: *D-stable sub-diffieties* $\leftrightarrow$ *θ-invariant sub-Higgs bundles.*

Caution: this is *not* an equivalence of moduli spaces.

Important observation

These constructions can be extended:

$$E \rightsquigarrow E^{\bullet} := [\cdots E^{i} \xrightarrow{d} E^{i+1} \xrightarrow{d} \cdots],$$

from bundles E to complexes $E^{\bullet}$ or more general objects of the derived category $\mathbf{D}^{b}(X)$.

Spencer-stability as PDE-stability provides a criteria for existence of metrics on these objects!

Set-theoretic Spencer–NAH II

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$\mathcal{D}$-Quot DG-Scheme

Natural to ask:

Question

Is there a derived enhancement $\mathbb{R}\mathcal{Q}uot_{\mathcal{D}_X}$? Is it representable?

Use a natural derived enhancement to $\mathcal{D}$-geometry:

$$\text{Algebraic } \mathcal{D}\text{-Geometry} \rightsquigarrow \text{Derived } \mathcal{D}\text{-Geometry}$$

$$\mathrm{Sch}_X(\mathcal{D}_X) \subset \mathrm{Fun}(\mathrm{CAlg}_X(\mathcal{D}_X)^{op}, \mathrm{Sets}) \rightsquigarrow \mathbf{dStk}_X(\mathcal{D}_X) \subset \mathbf{dPStk}_X(\mathcal{D}_X).$$

In a paraphrased form the main result of Part II [KSh2]

Theorem (K-S, arXiv:2411.02387, (2025))

Consider Theorem 1. It has a natural derived enhancement given by the $\mathcal{D}$-Quot functor $\mathbf{Quot}_{\mathcal{D}_X,Z}$. It is a $\mathcal{D}$-simplicial presheaf (satisfying descent) which is moreover representable by a dg-$\mathcal{D}$-manifold of $\mathcal{D}$-finite presentation.

Realizes an underlying classical finite-type $\mathcal{D}$-scheme as the classical truncation of a dg-$\mathcal{D}$-manifold obtained as a simplicial diagram of dg-schemes modulo actions of Lie pseudogroups (the $\mathcal{D}$-Quot dg-manifold).

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Idea of proof

Solve the derived moduli problem: selecting $m_0 \in \mathbb{N}$ large enough to ensure involutivity of $\mathcal{Y}$, we want to classify A_∞-submodules in a certain category of $\mathcal{D}_X$-modules, of the form

$$\mathcal{N}_{\geq m_0} \hookrightarrow \mathcal{C}h_{\geq m_0}, \quad with \;\; \dim(\mathcal{N}_k) = \mathbf{h}^{\mathcal{D}}(k), \forall k \geq m_0.$$

Rmk. Having a (graded) submodule $\mathcal{N}_{a,t} \hookrightarrow \mathcal{C}h$ is expressed via algebra relations on a product,

$$\mathrm{Gr}_{[a,t]} := \prod_{a \leq s \leq t} \mathrm{Gr}\big(\dim(\mathcal{N}_s), \mathcal{C}h_s\big).$$

N.B! Since symbol/characteristic modules are finitely-cogenerated if the equation is involutive and formally integrable.

We may can choose a basis in $\mathcal{N}_s \hookrightarrow \mathcal{C}h$, and this generates actions of Gl of rank $P(s)$ on $\mathcal{N}_s$.

Technical remarks

Solve in a 'stretch' $[a,t]$: extend beyond via MC-equations.

Based on constructing a dg-Lie algebra object in $\mathcal{D}$-modules,

$$\mathfrak{g}^{n}_{\geq a;t} := \mathcal{P}^{*}_{[n]\sqcup *}\big(\{\mathcal{A},\mathcal{N}_{\geq a}\};\mathcal{N}_t\big) \oplus \mathcal{P}^{*}_{[n]\sqcup\{*\}}\big(\{\mathcal{A},\mathcal{N}_{\geq a}\};\mathcal{C}h_t\big),$$

then, roughly speaking, one proceeds as follows:

- Use Koszul duality $\rightsquigarrow$ dg-commutative algebra object e.g. generated by $\mathfrak{g}^{\circ}[-1]$ leading to a (formal derived $\mathcal{D}$-stack)

$$\mathcal{Y}_{[a,a+2]} := \underline{\mathbf{Spec}}_{\mathcal{D}}\big(CE^{\bullet,\otimes^{\star}}(\mathfrak{g}^{\bullet}_{a,t}[1])^{\circ}\big). \tag{17}$$

  We end up with a sequence of formal derived $\mathcal{D}$-stacks,

$$\ldots \to \mathcal{Y}_{[a,a+2]} \to \mathcal{Y}_{[a,a+1]} \to \mathcal{Y}_a, \tag{18}$$

- Look at the 'injectivity' locus (so sub-module structures, not just A_{∞}-morphisms)
- Take a geometric quotient and computing a colimit (reduction to a $\mathcal{D}$-geometric Postnikov-type sequence).

Thank you for your attention!

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